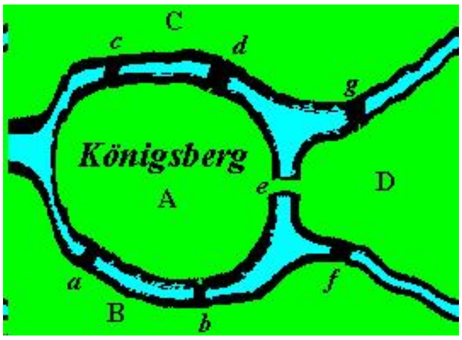
08. First and second degree equations

GRAPH THEORY

Based on Chris K. Caldwell work:
http://primes.utm.edu/cgi-bin/caldwell/tutor/graph/index.html

INTRODUCTION



Consider the next problem:

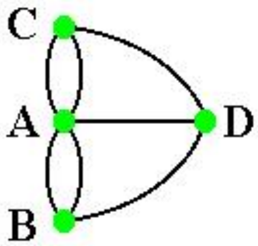
Old Königsberg has seven bridges (marked *a* through *g* in the sketch).

The townspeople wonder if it is possible to take a walk around the town in such a way as to cross each of the seven bridges exactly once.

What do you think? Try it?

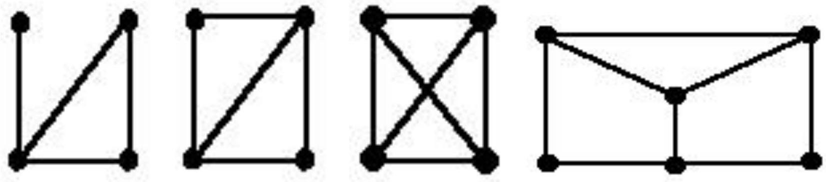
DEFINITION OF GRAPH

We could reduce each land mass of Königsberg to a single dot (which we will call a **vertex** or node), connected by arcs –the bridges– that we will call **edges**. This type of object (drawn on the left) is called a **graph**.



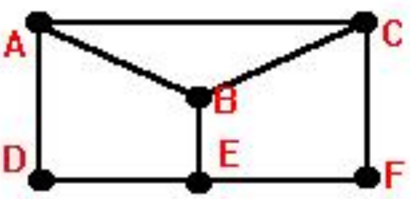
The Königsberg question is now: can we find a path around this graph that uses each edge exactly once?

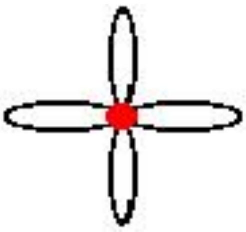
Here are a few more examples of graphs:



VERTICES: ADJACENCY AND DEGREE

Two vertices are **adjacent** if they are connected by an edge. In this graph vertex A is adjacent to B, C and D; but not adjacent to E or F.



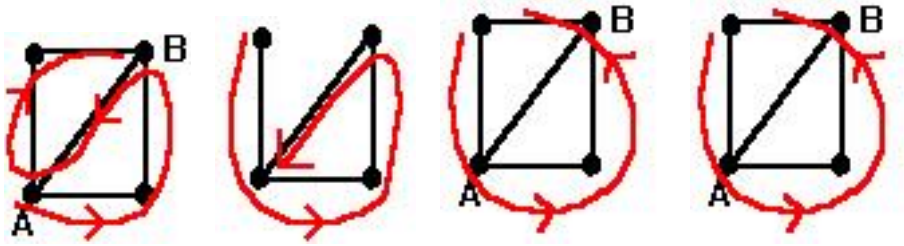


The **degree** of a vertex is the number of edge **ends** at that vertex. In the right graph the vertices A, B, C, and E have degree 3. Vertices D and F have degree 2.

So **a loop** (an edge that connects a vertex to itself) is degree 2. The vertex on the left has a degree of eight.

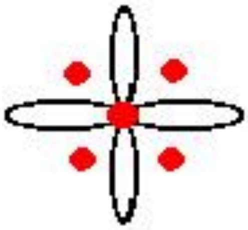
EDGES: PATHS, CIRCUITS, AND CONNECTIVITY

A **path** is a connected sequence of edges (connecting vertices) in a graph and the length of the path is the number of edges traversed. Below there are some examples of paths:



(The first has length five; the other three have length four).

A **circuit** is a path which ends at the vertex it begins, so a loop is a circuit of length one. Of the four paths shown above, only the last two are circuits.

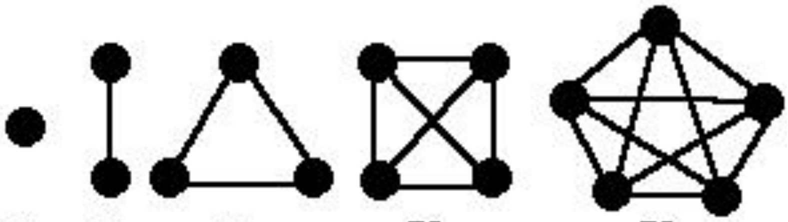


Finally, a graph is **connected** if there is a path connecting every pair of vertices. That is, a way to go from a vertex to another.

For example, this graph with five red vertices and black edges is not connected, in fact it is made of five separate **components** (five separate connected subgraphs: four vertices and one vertex with four loops).

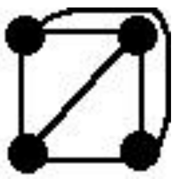
PLANAR GRAPHS

A graph is **planar** if *it can be drawn* (on a plane) so that the edges intersect only at the vertices. Note that it does not matter *how it is drawn*! It is planar if it is possible to draw it on a plane without edges crossing. For example, consider the first five complete graphs:



K_1 K_2 K_3 K_4 K_5

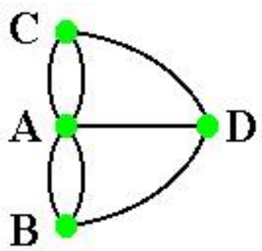
The first three are obviously planar, as is the fourth, since it *can be drawn* without the 'diagonals' intersecting:



It is planar if it can be drawn without edges crossing.

But what is about the fifth? Can it be drawn on the plane without intersecting edges? Below is one attempt to draw it, one intersection still remains. Get out some paper and see if you can do better.

EULER PATHS AND CIRCUITS



An **Euler path** is a path which traverses every edge in a graph exactly once.

> **Theorem:**
> A connected graph has an Euler path (which is not a circuit) if and only if it has exactly **two** vertices with **odd** degree.

An **Euler circuit** is an Euler path that is a circuit.

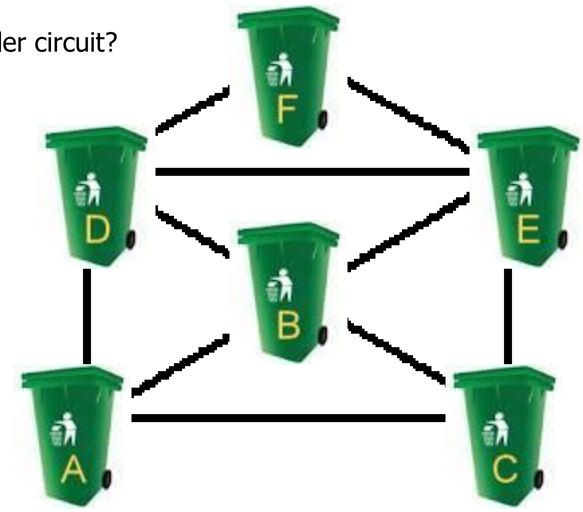
> **Theorem:**
> A connected graph has an Euler circuit if and only if **every** vertex has **even** degree.

Reason: An Euler circuit is an Euler path so again the points that are not endpoints must have even degree (twice the number of times the circuit passes through). Unlike a path, the circuit endpoints also have even degree: **twice** the number of times the circuit passes through **plus one** for the start of the circuit **and one** for the end.

APPLICATIONS

A garbage truck has to collect the rubbish by travelling through the points you see, the lines are the roads that connect these.

Is there any possible Euler path? And an Euler circuit?



1. EQUATIONS

PARTS OF AN ALGEBRAIC EXPRESSION

Variables: They are the letters.

Terms: Each adding part.

Coefficients: The numbers which are multiplying the variables.

Independent term: The number without letter.

Numerical value of an algebraic expression: The value that results of replacing the variable by a number.

Write for $2x^2 - 3x - 4$:

a) Terms:

b) Coeficients:

c) Variable:

d) Numerical value for x = 1

e) Numerical value for x = –2.

EQUATIONS

Algebra is the section of mathematics that studies symbolic language.

For example, the area formula of a rectangle: A = b · a

Two consecutive numbers add up to 15: x + x +1 = 15.

An equation is an equality with unknowns. We describe the unknown values by letters. The equation expresses the numeric relation between the data and the unknowns.

For example,

a) 4x – 5 = 7; b) x^2 + x – 2 = 0; c) x + y = 7

Solution is any value that makes the ____________ right.

Could you write some solution for each previous equation? Check your answers.

a)

b)

c)

To solve an equation is the procedure that we follow to find the value of the unknown. Equivalent equations have the same solutions.

Write an equivalent equation to $3x + 5 = 11$

TRANSPOSITION RULES

These are the rules that we use to get equivalent equations. Write on the right the results after using the property. The first one is done as example.

1. A number which is adding to a member can be removed by subtracting this quantity in both members.	$x + 4 = 7$	$x = 7 - 4$
2. A number which is subtracting from a member can be removed by adding this quantity in both members.	$3x - 5 = 12$	
3. A number which is multiplying a member can be removed by dividing this quantity in both members.	$3 \cdot (x - 6) = 9$	
4. A number which is dividing a member can be removed by multiplying this quantity in both members.	$\frac{5x - 8}{3} = 6$	
5. We can change the sign of both members together.	$-8x + 5 = 2x - 9$	
6. If a number is operating in the same way in two members we can simplify it.	a) $5x - \cancel{8} = 6x - \cancel{8}$ b) $(2x + 3) \cdot \cancel{4} = (-2 + x) \cdot \cancel{4}$ c) $\frac{x + 3}{\cancel{7}} = \frac{2x - 9}{\cancel{7}}$	

2. FIRST–DEGREE EQUATION

They are equations whose highest power for the unknown is one.

RESOLUTION OF EQUATIONS WITH WHOLE COEFFICIENTS

To solve this kind of equations we have to do the following steps:

1. Operating parenthesis.
2. Grouping like terms.
3. Clearing x.
4. Checking the solution.

RESOLUTION OF EQUATIONS WITH DENOMINATORS

- To solve this kind of equations we have to do the following steps:
  1. Removing the denominators.
  2. Operating parenthesis.
  3. Grouping like terms.
  4. Clearing x.
  5. Checking the solution.

RESOLUTION OF PROBLEMS WITH EQUATIONS

In science we usually know some information about the unknown values that we want to find. These are the data.

Solving the problem is to find the value of the unknown quantities using the given information.

How to solve them?

You must follow these four steps:

1. Locating the unknown and calling it x.
2. Writing the information in terms of it.
3. Solving the equation.
4. Writing the solution.
5. Checking the solution.

3. SECOND DEGREE EQUATIONS. QUADRATIC EQUATIONS

A quadratic equation is an equation whose unknown is second degree. Therefore, it has three kinds of coefficients: second degree; first degree and independent.

For example, $a) -7x^2+5x-8=2x+6; \quad b)\ 4x-3x^2+5=8x^2-3x+6$

We can always reduce a quadratic equation to an expression like this:

$ax^2+bx+c=0$. This is the general form.

Could you reduce the above equations to the general form?

a)

b)

A medieval mathematician discovers the way to isolate the unknown for these kinds of equations and he got the next formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\rightarrow\begin{cases}x=\dfrac{-b+\sqrt{b^2-4ac}}{2a}\\ x=\dfrac{-b-\sqrt{b^2-4ac}}{2a}\end{cases}$$

Solve the equation: $2x^2+6x+4=0$

INCOMPLETE EQUATIONS

These are the quadratic equations without some of the terms

Type 1. If the equation doesn't have first–degree term we will clear the unknown directly.

For example, solve the equation: $2x^2 - 8 = 0$

And now, $(x-1)^2 - 9 = 0$

Type 2. If the equation doesn't have independent term we will solve the equation by factorization. For this sort of equations we always get 0 as solution.

For example, solve the equation: $2x^2 + 8x = 0$.

And now, solve: $-3x^2 + 6x = 0$

EXERCISES AND PROBLEMS

1. EQUATIONS

2. FIRST–DEGREE EQUATION

1. Solve the following equations:

a) 2x = 6; b) 2x – 3 = 6 + x; c) 2(2x – 3) = 6 + x

2. Solve: 2 (x+1) – 3 (x – 2) = x + 6

3. Solve the equation: 4 (x – 10) = – 6 (2 – x) – 6x

4. Solve the following equations:

$$a)\ \frac{x-1}{6}-\frac{x-3}{2}=-1;\ \ b)\ \frac{3}{4}\left(2x+4\right)=x+19$$

5. Solve the following equation: $\frac{x-1}{4}-\frac{x-5}{36}=\frac{x+5}{9}$

6. Solve the following equations:

a) $\frac{x-1}{6}-\frac{x-3}{2}=-1$; b) $2(x+1)-3(x-2)=x+6$

7. Solve the following equations:

a) $\frac{3x+1}{7}-\frac{2-4x}{3}=\frac{-5x-4}{14}+\frac{7x}{6}$; b) $\frac{5}{x-7}=\frac{3}{x-2}$ (cross multiply)

8. Solve the following equation: $\frac{x-1}{2}-\frac{x+1}{3}=x-\frac{5}{2}$

9. Solve the following equation: $\frac{x+2}{3}-\frac{3x-4}{4}=2x-8$

10. Solve the equation: $\frac{x-1}{6}-\frac{x-3}{2}=-1$

11. At a community meeting there are double the number of women to men and triple the number of children to the total of men and women combined. How many men, women and children are there if 96 people attend the meeting?

12. We remove 7/8 of a tank of oil. Then, we put 38 litres into the tank and thus the tank is 3/5 full of its capacity. Calculate the total capacity of the tank.

13. In a bookshop, Anne buys a novel with a third of her money and a comic with two–thirds of what remained. When she left the bookstore, she had €12. How much money did Anne have before arriving at the store?

14. A clock sounds to indicate the time of 3 o'clock. a) At what time between 3 and 4 will the hour and minute clock hands overlap? b) At what time will its clock hands form a right angle for the first time?

15. The printing company charges €20 a book and the bookshop charges a 30% of its final price. What price does a writer have to sell a book to make a profit 10 € per book?

16. A seller wants to sell each product for 1 €. If he has to add a 18% of V.A.T. and a 20% of profit, how much does the factory price have to be?

17. The ages of a mother and her son add up to 40 years. After 14 years the mother's age will be triple that of the son. What are the ages of each one today?

18. The ages of a mother and her son add up to 43 years. After 4 years the mother's age will be double that of the son. What is the age of each one today?

19. Fill in the gaps with numbers so the third number is the sum of the first and the second, the fourth is the sum of the second and the third and so on. And finally the last number is 81.

5						81

20. Today, Ann's age is triple that of her daughter's age, and 5 years ago it was five times as much. What is the age of each one today?

21. A mother is 35 years older than her son and after 15 years her age will be double than the son. What are the ages of each one today?

3. SECOND DEGREE EQUATIONS. QUADRATIC EQUATIONS

22. Calculate the unknown in each case by clearing the unknown step by step, that is, without using any formula:

a) $\frac{3x^2-8}{2}=2$ *b)* $\frac{(x+3)^2}{3}-1=2$ *c)* $\left(\frac{x}{2}\right)^2+4=5$

23. Solve the following equations: $x^2 - 8 = 0$

24. Solve the quadratic equations using the suitable method for each:
a) $4x^2-16=0$; b) $6x^2+3x=0$

25. Solve the following equation: $x^2-4x=0$

26. Solve the following equation: $x^2 + 2x - 15=0$

27. Solve the following equation: $3x^2 - 7x + 2 = 0$

28. Solve the following equation: $5x^2 - 6x - 8 = 0$

29. Solve the following equation: $2x-3=1-2x+x^2$

30. Solve the following equation: $4x - 6 = 2 - 4x + 2x^2$

31. Solve the equation: $18=6x+x(x-13)$

32. Solve the following equations: a) $x^2+\left(x+2\right)^2=580$; b) $x^2-5x-84=0$

33. Solve the following equation: $3x^2-\frac{3x}{4}-\frac{9}{8}=0$

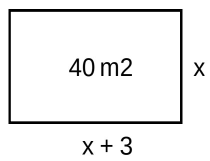
34. Solve the following equations:
a) $4x^2-6x+2=0$; b) $x^2-\frac{7}{6}x+\frac{1}{3}=0$ (first, remove the denominators)

35. Determine the quadratic equation whose solutions are: 3 and −2.

36. Write the general form of a second degree equation whose solutions are −5 and 7.

37. The sum of two numbers is 5 and their product is −84. Find these numbers.

38. This rectangle is 40 m^2. How long are its sides? Write the equation and solve it.



39. To fence a rectangular farm of 750 m^2, 110 m of fence has to be used. Calculate the dimensions of the farm.

40. A rectangular garden 50 m long and 34 m wide is surrounded by a uniform dirt road. Find the width of the road if the total area of the garden and road is 5400 m^2.

41. The lengths of the sides of a right–angled triangle are measured as three consecutive even numbers (in cm). Find the values of these sides. You have to use the Pythagorean Theorem to solve it.

42. Find the side of a square knowing that if we add 5 cm to two parallel sides we get a 24 cm^2 rectangle.